

Integral and series transformations via Ramanujan's identities and Salem's type equivalences to the Riemann hypothesis

Semyon YAKUBOVICH

Abstract

We consider integral and series transformations, which are associated with Ramanujan's identities, involving arithmetic functions $a(n), \omega(n), \sigma_a(n), d(n), \mu(n), \lambda(n), \varphi(n)$ and a ratio of products of Riemann's zeta functions of different arguments. Reciprocal inversion formulas are proved in a Banach space of functions whose Mellin's transforms are integrable over the vertical line $\operatorname{Re} s > 1$. Examples of new transformations like Widder-Lambert and Kontorovich-Lebedev type are exhibited. Particular cases include familiar Lambert and Möbius transformations. Finally a class of equivalences of the Salem type to the Riemann hypothesis is established.

Keywords: *Mellin transform, Riemann zeta-function, Kontorovich-Lebedev transform, modified Bessel functions, Lambert transform, Möbius transform, Ramanujan's formulas, arithmetic functions, Lambert series, the Riemann hypothesis*

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1 Introduction and auxiliary results

Integral and series transformations, which will be derived in the sequel are based on remarkable Ramanujan's identities involving arithmetic and Riemann's zeta-functions [6], [10], namely

$$\frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} = \sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s}, \quad (1.1)$$

where $\operatorname{Re} s > \max\{1, \operatorname{Re} a + 1, \operatorname{Re} b + 1, \operatorname{Re} (a + b) + 1\}$,

$$\zeta(s)\zeta(s-a) = \sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s}, \quad (1.2)$$

where $\operatorname{Re} s > \max\{1, \operatorname{Re} a + 1\}$,

$$\frac{\zeta^2(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{2^{\omega(n)}}{n^s}, \quad \operatorname{Re} s > 1, \quad (1.3)$$

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}, \quad \operatorname{Re} s > 1, \quad (1.4)$$

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad \operatorname{Re} s > 1, \quad (1.5)$$

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s}, \quad \text{Res} > 1, \quad (1.6)$$

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \quad \text{Res} > 1, \quad (1.7)$$

$$\frac{\zeta^3(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d(n^2)}{n^s}, \quad \text{Res} > 1, \quad (1.8)$$

$$\frac{\zeta^4(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{d^2(n)}{n^s}, \quad \text{Res} > 1, \quad (1.9)$$

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^s}, \quad \text{Res} > 2, \quad (1.10)$$

$$\frac{1-2^{1-s}}{1-2^{-s}} \zeta(s-1) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \quad \text{Res} > 2. \quad (1.11)$$

Here $\zeta(s)$ is the Riemann zeta-function [10], which satisfies the familiar functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s), \quad (1.12)$$

where $\Gamma(z)$ is Euler's gamma-function, and in the half-plane $\text{Res} = c_0 > 1$ it is represented by the absolutely and uniformly convergent series with respect to $t \in \mathbb{R}$, $s = c_0 + it$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad (1.13)$$

and by the uniformly convergent series

$$(1-2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}, \quad \text{Res} > 0. \quad (1.14)$$

Further, $a(n)$ in (1.11) denotes the greatest odd divisor of n , $\sigma_a(n)$ in (1.1), (1.2) is the sum of a -th powers of the divisors of $n \in \mathbb{N}$. In particular, for pure imaginary $a = i\tau$ $|\sigma_{i\tau}(n)| \leq d(n)$, where $d(n)$ is the Dirichlet divisor function, i.e. the number of divisors of n , including 1 and n itself. It has the estimate [10] $d(n) = O(n^\varepsilon)$, $n \rightarrow \infty$, $\varepsilon > 0$. The Möbius function is denoted by $\mu(n)$ and $|\mu(n)| \leq 1$. The symbol $\omega(n)$ in (1.3) represents the number of distinct prime factors of n and it behaves as $\omega(n) = O(\log \log n)$, $n \rightarrow \infty$ (see in [8]). By $\varphi(n)$ Euler's totient function is denoted and its asymptotic behavior satisfies [cf. [8]] $\varphi(n) = O(n[\log \log n]^{-1})$, $n \rightarrow \infty$. Finally, $\lambda(n)$ in (1.7) is the Liouville function, $|\lambda(n)| \leq 1$.

Following similar ideas presented in [12], [14] we define a special functional space $\mathcal{M}^{-1}(L_c)$, which will be suitable for our investigation of the series transformations with arithmetic functions.

Definition 1. Denote by $\mathcal{M}^{-1}(L_c)$ the space of functions $f(x), x \in \mathbb{R}_+$, representable by inverse Mellin transform of integrable functions $f^*(s) \in L_1(c)$ on the vertical line $c = \{s \in \mathbb{C} : \text{Res} = c_0\}$:

$$f(x) = \frac{1}{2\pi i} \int_c f^*(s) x^{-s} ds. \quad (1.15)$$

The space $\mathcal{M}^{-1}(L_c)$ with the usual operations of addition and multiplication by scalar is a linear vector space. If the norm in $\mathcal{M}^{-1}(L_c)$ is introduced by the formula

$$\|f\|_{\mathcal{M}^{-1}(L_c)} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |f^*(c_0 + it)| dt, \quad (1.16)$$

then it becomes a Banach space. Simple properties of the space $\mathcal{M}^{-1}(L_c)$ follow immediately from Definition 1 and the basic properties of the Fourier and Mellin transforms of integrable functions. For instance, the Riemann-Lebesgue lemma yields that $x^{c_0} f(x)$ is uniformly bounded, continuous on \mathbb{R}_+ and $x^{c_0} f(x) = o(1)$, when $x \rightarrow +\infty$ and $x \rightarrow 0$. Moreover, if $f(x), g(x) \in \mathcal{M}^{-1}(L_c)$, where $g(x)$ is the inverse Mellin transform (1.14) of the function $g^*(s)$, then $x^{c_0} f(x)g(x) \in \mathcal{M}^{-1}(L_c)$ because the product $x^{c_0} f(x)g(x)$ is the inverse Mellin transform of the function

$$\frac{1}{2\pi i} \int_c f^*(\tau) g^*(s - \tau + c_0) d\tau,$$

which belongs to $L_1(c)$ by Fubini's theorem. Finally we note that if $f(x) \in \mathcal{M}^{-1}(L_c)$ and $x^{c_0-1} g(x) \in L_1(\mathbb{R}_+)$, then the Mellin convolution

$$\int_0^\infty g(u) f\left(\frac{x}{u}\right) \frac{du}{u} \in \mathcal{M}^{-1}(L_c).$$

In fact, the latter integral is an inverse Mellin transform of the function $f^*(s)g^*(s)$ and since $f^*(s) \in L_1(c)$ and $g^*(s)$ is essentially bounded on c , we have $f^*(s)g^*(s) \in L_1(c)$.

A more general space $\mathcal{M}_{c_1, c_2}^{-1}(L_c)$, which will be involved as well is defined similarly to the one in [12], [14].

Definition 2. Let $c_1, c_2 \in \mathbb{R}$ be such that $2\text{sign } c_1 + \text{sign } c_2 \geq 0$. By $\mathcal{M}_{c_1, c_2}^{-1}(L_c)$ we denote the space of functions $f(x), x \in \mathbb{R}_+$, representable in the form (1.15), where $s^{c_2} e^{\pi c_1 |s|} f^*(s) \in L_1(c)$.

It is a Banach space with the norm

$$\|f\|_{\mathcal{M}_{c_1, c_2}^{-1}(L_c)} = \frac{1}{2\pi} \int_c e^{\pi c_1 |s|} |s^{c_2} f^*(s)| ds, \quad \text{Res} = c_0.$$

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2 Transformations with arithmetic functions

We begin with the following result.

Theorem 1. Let $f \in \mathcal{M}^{-1}(L_c)$, $c_0 > 1$. Then for all $x > 0$ the following series expansions with the Möbius function are true

$$f(x) = \sum_{n=1}^{\infty} \mu(n) \sum_{m=1}^{\infty} f(xnm), \quad (2.1)$$

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mu(m) f(xnm), \quad (2.2)$$

$$f(x) = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} 2^k \mu(n) \sum_{m=1}^{\infty} (-1)^{m-1} f(xnm2^k), \quad (2.3)$$

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} 2^k \mu(m) f(xnm2^k), \quad (2.4)$$

$$f(x) - 2f(2x) = \sum_{n=1}^{\infty} \mu(n) \sum_{m=1}^{\infty} (-1)^{m-1} f(xnm). \quad (2.5)$$

Moreover, expansions (2.1), (2.2) generate reciprocal pair of transformations

$$g(x) = \frac{1}{2\pi i} \int_c \zeta(s) f^*(s) x^{-s} ds = \sum_{n=1}^{\infty} f(xn), \quad (2.6)$$

$$f(x) = \frac{1}{2\pi i} \int_c \frac{g^*(s)}{\zeta(s)} x^{-s} ds = \sum_{n=1}^{\infty} \mu(n) g(xn), \quad (2.7)$$

which are automorphisms of the space $\mathcal{M}^{-1}(L_c)$ and satisfy the following inequalities for the norms

$$[\zeta(c_0)]^{-1} \|g\|_{\mathcal{M}^{-1}(L_c)} \leq \|f\|_{\mathcal{M}^{-1}(L_c)} \leq \zeta(c_0) \|g\|_{\mathcal{M}^{-1}(L_c)}, \quad c_0 > 1. \quad (2.8)$$

Analogously expansions (2.3), (2.4) generate reciprocal transformations

$$g(x) = \frac{1}{2\pi i} \int_c (1 - 2^{1-s}) \zeta(s) f^*(s) x^{-s} ds = \sum_{n=1}^{\infty} (-1)^{n-1} f(xn), \quad (2.9)$$

$$f(x) = \frac{1}{2\pi i} \int_c \frac{g^*(s)}{(1 - 2^{1-s}) \zeta(s)} x^{-s} ds = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} 2^k \mu(n) g(xn2^k), \quad (2.10)$$

which are automorphisms of the space $\mathcal{M}^{-1}(L_c)$ and satisfy the norm estimates

$$[\zeta(c_0)]^{-1} \|g\|_{\mathcal{M}^{-1}(L_c)} \leq \|f\|_{\mathcal{M}^{-1}(L_c)} \leq (1 - 2^{1-c_0})^{-1} \zeta(c_0) \|g\|_{\mathcal{M}^{-1}(L_c)}, \quad c_0 > 1. \quad (2.11)$$

Proof. In fact, the validity of equalities (2.1)- (2.5) follows immediately from representation (1.15), identities (1.5), (1.13), (1.14) and elementary sum of geometric progression after the change of the order of summation and integration via Fubini's theorem owing to the following estimates

$$\begin{aligned} \sum_{n=1}^{\infty} |\mu(n)| \sum_{m=1}^{\infty} |f(xnm)| &\leq \frac{x^{-c_0} \zeta^2(c_0)}{2\pi} \int_c |f^*(s) ds| < \infty, \quad x > 0, \\ \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} 2^k |\mu(n)| \sum_{m=1}^{\infty} |f(xnm2^k)| &\leq \frac{x^{-c_0} \zeta^2(c_0)}{2\pi(1 - 2^{1-c_0})} \int_c |f^*(s) ds| < \infty, \quad x > 0. \end{aligned}$$

Hence we establish reciprocal equalities (2.6), (2.7), where $f^*(s) = g^*(s)[\zeta(s)]^{-1}$ by virtue of the uniqueness theorem for the Mellin transform. This also guarantees the automorphism of the space $\mathcal{M}^{-1}(L_c)$ under transformations (2.6), (2.7). Finally, since (see (1.16))

$$\|f\|_{\mathcal{M}^{-1}(L_c)} = \frac{1}{2\pi} \int_c \left| \zeta(s) f^*(s) \frac{ds}{\zeta(s)} \right| \leq \zeta(c_0) \|g\|_{\mathcal{M}^{-1}(L_c)},$$

$$\|g\|_{\mathcal{M}^{-1}(L_c)} = \frac{1}{2\pi} \int_c |\zeta(s) f^*(s) ds| \leq \zeta(c_0) \|f\|_{\mathcal{M}^{-1}(L_c)},$$

we prove inequalities (2.8). In the same manner we establish the automorphism of the space $\mathcal{M}^{-1}(L_c)$ under reciprocal pair (2.9), (2.10) and estimates ($g^*(s) = f^*(s)\zeta(s)(1 - 2^{1-s})$)

$$\|g\|_{\mathcal{M}^{-1}(L_c)} = \frac{1}{2\pi} \int_c |(1 - 2^{1-s}) \zeta(s) f^*(s) ds| \leq \zeta(c_0) \|f\|_{\mathcal{M}^{-1}(L_c)},$$

$$\|f\|_{\mathcal{M}^{-1}(L_c)} = \frac{1}{2\pi} \int_c \left| \frac{g^*(s)ds}{(1-2^{1-s})\zeta(s)} \right| \leq (1-2^{1-c_0})^{-1} \zeta(c_0) \|g\|_{\mathcal{M}^{-1}(L_c)},$$

yield inequalities (2.11). \square

Analogously, calling Ramanujan's identities (1.2)- (1.10) we come out with two more theorems, which we leave without proof.

Theorem 2. *Let $f \in \mathcal{M}^{-1}(L_c)$, $c_0 > 1$. Then for all $x > 0$ the following series expansions with arithmetic functions hold valid*

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} |\mu(n)| \sum_{m=1}^{\infty} \lambda(m) f(xnm), \\ f(x) &= \sum_{n=1}^{\infty} \lambda(n) \sum_{m=1}^{\infty} |\mu(m)| f(xnm), \\ f(x) &= \sum_{k,n=1}^{\infty} \mu(k) \lambda(n) \sum_{m=1}^{\infty} 2^{\omega(m)} f(xnmk), \\ f(x) &= \sum_{n=1}^{\infty} 2^{\omega(n)} \sum_{k,m=1}^{\infty} \lambda(k) \mu(m) f(xnmk), \\ f(x) &= \sum_{k,m=1}^{\infty} \mu(k) \mu(m) \sum_{n=1}^{\infty} d(n) f(xnmk), \\ f(x) &= \sum_{m=1}^{\infty} d(m) \sum_{k,n=1}^{\infty} \mu(k) \mu(n) f(xnmk), \\ f(x) &= \sum_{k,n,j=1}^{\infty} \mu(k) \mu(j) \lambda(n) \sum_{m=1}^{\infty} d(m^2) f(xnmjk), \\ f(x) &= \sum_{m=1}^{\infty} d(m^2) \sum_{k,n,j=1}^{\infty} \mu(k) \mu(n) \lambda(j) f(xnmkj), \\ f(x) &= \sum_{k,n,j,l=1}^{\infty} \mu(k) \mu(j) \mu(l) \lambda(n) \sum_{m=1}^{\infty} d^2(m) f(xnmjkl), \\ f(x) &= \sum_{m=1}^{\infty} d^2(m) \sum_{k,n,j,l=1}^{\infty} \mu(k) \mu(n) \mu(l) \lambda(j) f(xnmkjl), \\ f(x) &= \sum_{k,n=1}^{\infty} k \mu(k) \sum_{m=1}^{\infty} \varphi(m) f(xnmk), \quad c_0 > 2, \\ f(x) &= \sum_{m=1}^{\infty} \varphi(m) \sum_{k,n=1}^{\infty} k \mu(k) f(xnmk), \quad c_0 > 2, \\ f(x) &= \sum_{k,n=1}^{\infty} n^a \mu(k) \mu(n) \sum_{m=1}^{\infty} \sigma_a(m) f(xnmk), \quad c_0 > \max\{1, \operatorname{Re} a + 1\}, \end{aligned}$$

$$f(x) = \sum_{m=1}^{\infty} \sigma_a(m) \sum_{k,n=1}^{\infty} k^a \mu(k) \mu(n) f(xnmk), \quad c_0 > \max\{1, \operatorname{Re} a + 1\}.$$

Theorem 3. *Let $f \in \mathcal{M}^{-1}(L_c)$, $c_0 > 1$. Then for all $x > 0$ the following reciprocal series transformations are automorphisms in $\mathcal{M}^{-1}(L_c)$ with the corresponding norm estimates, namely*

$$g(x) = \sum_{m=1}^{\infty} \lambda(m) f(xm),$$

$$f(x) = \sum_{n=1}^{\infty} |\mu(n)| g(xn),$$

$$[\zeta(c_0)]^{-1} \|g\|_{\mathcal{M}^{-1}(L_c)} \leq \|f\|_{\mathcal{M}^{-1}(L_c)} \leq \frac{\zeta(c_0)}{\zeta(2c_0)} \|g\|_{\mathcal{M}^{-1}(L_c)}, \quad c_0 > 1;$$

$$g(x) = \sum_{n=1}^{\infty} 2^{\omega(n)} f(xn),$$

$$f(x) = \sum_{k,m=1}^{\infty} \lambda(k) \mu(m) g(xmk),$$

$$\frac{\zeta(2c_0)}{\zeta^2(c_0)} \|g\|_{\mathcal{M}^{-1}(L_c)} \leq \|f\|_{\mathcal{M}^{-1}(L_c)} \leq \zeta^2(c_0) \|g\|_{\mathcal{M}^{-1}(L_c)}, \quad c_0 > 1;$$

$$g(x) = \sum_{n=1}^{\infty} d(n) f(xn),$$

$$f(x) = \sum_{k,n=1}^{\infty} \mu(k) \mu(n) g(xkn),$$

$$\zeta^{-2}(c_0) \|g\|_{\mathcal{M}^{-1}(L_c)} \leq \|f\|_{\mathcal{M}^{-1}(L_c)} \leq \zeta^2(c_0) \|g\|_{\mathcal{M}^{-1}(L_c)}, \quad c_0 > 1;$$

$$g(x) = \sum_{m=1}^{\infty} d(m^2) f(xm),$$

$$f(x) = \sum_{k,n,j=1}^{\infty} \mu(k) \mu(j) \lambda(n) g(xnjk),$$

$$\zeta^{-3}(c_0) \|g\|_{\mathcal{M}^{-1}(L_c)} \leq \|f\|_{\mathcal{M}^{-1}(L_c)} \leq \zeta^3(c_0) \zeta(2c_0) \|g\|_{\mathcal{M}^{-1}(L_c)}, \quad c_0 > 1;$$

$$g(x) = \sum_{m=1}^{\infty} d^2(m) f(xm),$$

$$f(x) = \sum_{k,n,j,l=1}^{\infty} \mu(k) \mu(j) \mu(l) \lambda(n) g(xnjk l),$$

$$\zeta^{-4}(c_0) \|g\|_{\mathcal{M}^{-1}(L_c)} \leq \|f\|_{\mathcal{M}^{-1}(L_c)} \leq \zeta^4(c_0) \zeta(2c_0) \|g\|_{\mathcal{M}^{-1}(L_c)}, \quad c_0 > 1;$$

$$g(x) = \sum_{m=1}^{\infty} \varphi(m) f(xm),$$

$$f(x) = \sum_{k,n=1}^{\infty} k\mu(k)g(xnk),$$

$$[\zeta(c_0)\zeta(c_0-1)]^{-1} \|g\|_{\mathcal{M}^{-1}(L_c)} \leq \|f\|_{\mathcal{M}^{-1}(L_c)} \leq \zeta(c_0)\zeta(c_0-1) \|g\|_{\mathcal{M}^{-1}(L_c)}, \quad c_0 > 2;$$

$$g(x) = \sum_{m=1}^{\infty} \sigma_a(m)f(xm),$$

$$f(x) = \sum_{k,n=1}^{\infty} n^a \mu(k)\mu(n)g(xnk),$$

$$[\zeta(c_0)\zeta(c_0 - \operatorname{Re} a)]^{-1} \|g\|_{\mathcal{M}^{-1}(L_c)} \leq \|f\|_{\mathcal{M}^{-1}(L_c)} \leq \zeta(c_0)\zeta(c_0 - \operatorname{Re} a) \|g\|_{\mathcal{M}^{-1}(L_c)}, \quad c_0 > \max\{1, \operatorname{Re} a + 1\}.$$

Let $f(x) = e^{-x}$, which evidently belongs to the space $\mathcal{M}^{-1}(L_c)$ since $f^*(s) = \Gamma(s)$ is the Euler gamma-function. Substituting it in (2.1)-(2.5) and calculating elementary series we come out to the Lambert type expansions (cf. [8])

$$e^{-x} = \sum_{n=1}^{\infty} \frac{\mu(n)}{e^{xn} - 1}, \quad x > 0,$$

$$e^{-x} = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{2^k \mu(n)}{\exp(xn2^k) + 1}, \quad x > 0,$$

$$e^{-x} - 2e^{-2x} = \sum_{n=1}^{\infty} \frac{\mu(n)}{e^{xn} + 1}, \quad x > 0.$$

Further, the Parseval equality for the Mellin transform [11] and Fubini's theorem allow to write the modified Laplace transform [3] of $f \in \mathcal{M}^{-1}(L_c)$ in the form

$$\int_0^{\infty} e^{-x/t} f(t) \frac{dt}{t} = \frac{1}{2\pi i} \int_c \Gamma(s) f^*(s) x^{-s} ds. \quad (2.12)$$

Moreover, due to Definition 2 and Stirling's asymptotic formula for gamma-functions [1, Vol. 1] it forms a bijective map of the space $\mathcal{M}^{-1}(L_c)$ onto its subspace $\mathcal{M}_{1/2, 1/2-c_0}^{-1}(L_c)$. Thus appealing to Theorem 1 we will derive the Widder type inversion formulas for the Lambert transform (see in [2], [4], [13]) and Widder-Lambert type transforms. Precisely, we prove

Theorem 4. *Let $f \in \mathcal{M}^{-1}(L_c)$ and $c_0 > 1$. Then the modified Lambert transform*

$$g(x) = \int_0^{\infty} \frac{f(t)dt}{t(e^{x/t} - 1)}, \quad x > 0 \quad (2.13)$$

maps bijectively onto the space $\mathcal{M}_{1/2, 1/2-c_0}^{-1}(L_c)$ and for all $x > 0$ the Widder type inversion formula holds true

$$f(x) = \lim_{k \rightarrow \infty} \left(-x \frac{d}{dx} \right) \prod_{j=1}^k \left(1 - \frac{x}{j} \frac{d}{dx} \right) \sum_{n=1}^{\infty} \mu(n) g(xkn). \quad (2.14)$$

Analogously, the Widder-Lambert type transformation

$$g(x) = \int_0^{\infty} \frac{f(t)dt}{t(e^{x/t} + 1)}, \quad x > 0 \quad (2.15)$$

is a bijective map between spaces $\mathcal{M}^{-1}(L_c)$, $\mathcal{M}_{1/2,1/2-c_0}^{-1}(L_c)$ and for all $x > 0$ the following inversion formula takes place

$$f(x) = \lim_{k \rightarrow \infty} \left(-x \frac{d}{dx} \right) \prod_{j=1}^k \left(1 - \frac{x}{j} \frac{d}{dx} \right) \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} 2^m \mu(n) g(xkn2^m). \quad (2.16)$$

Proof. In fact, the proof is based on Theorem 1, equality (2.12), a familiar infinite product for the gamma-function (see, for instance, [14, p.48])

$$\frac{1}{\Gamma(s)} = \lim_{k \rightarrow \infty} sk^{-s} \prod_{j=1}^k \left(1 + \frac{s}{j} \right),$$

and the asymptotic behavior $|\Gamma(s)|^{-1} = e^{\pi|s|/2}|s|^{1/2-c_0}$, $s = c_0 + it$, $|t| \rightarrow \infty$ via Stirling formula. So owing to Theorem 1 and the absolute and uniform convergence, which guarantees the change of the order of integration and summation, the modified Lambert transform bijectively maps $\mathcal{M}^{-1}(L_c)$ onto $\mathcal{M}_{1/2,1/2-c_0}^{-1}(L_c)$ and represented by (2.13), namely

$$\begin{aligned} g(x) &= \frac{1}{2\pi i} \int_c \zeta(s) \Gamma(s) f^*(s) x^{-s} ds = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-xn/t} f(t) \frac{dt}{t} \\ &= \int_0^{\infty} \frac{f(t) dt}{t(e^{x/t} - 1)}, x > 0. \end{aligned}$$

Reciprocally, following similarly to [14, p.49] and appealing to the Lebesgue dominated convergence theorem and equality (2.7), we find

$$\begin{aligned} f(x) &= \frac{1}{2\pi i} \int_c \frac{g^*(s) x^{-s}}{\zeta(s) \Gamma(s)} ds = \lim_{k \rightarrow \infty} \frac{1}{2\pi i} \int_c \prod_{j=1}^k \left(1 + \frac{s}{j} \right) \frac{sg^*(s)(kx)^{-s}}{\zeta(s)} ds \\ &= \lim_{k \rightarrow \infty} \left(-x \frac{d}{dx} \right) \prod_{j=1}^k \left(1 - \frac{x}{j} \frac{d}{dx} \right) \sum_{n=1}^{\infty} \mu(n) g(xkn), \end{aligned}$$

which gives (2.14). In the same manner, employing again (1.14) and Theorem 1 we deduce the representation (2.15) of the Widder-Lambert type transform

$$\begin{aligned} g(x) &= \frac{1}{2\pi i} \int_c (1 - 2^{1-s}) \zeta(s) \Gamma(s) f^*(s) x^{-s} ds = \sum_{n=1}^{\infty} (-1)^{n-1} \int_0^{\infty} e^{-xn/t} f(t) \frac{dt}{t} \\ &= \int_0^{\infty} \frac{f(t) dt}{t(e^{x/t} + 1)}, x > 0. \end{aligned}$$

Finally, the same motivations perform the chain of equalities

$$\begin{aligned} f(x) &= \frac{1}{2\pi i} \int_c \frac{g^*(s) x^{-s}}{(1 - 2^{1-s}) \zeta(s) \Gamma(s)} ds = \lim_{k \rightarrow \infty} \frac{1}{2\pi i} \sum_{m=0}^{\infty} 2^m \int_c \prod_{j=1}^k \left(1 + \frac{s}{j} \right) \frac{sg^*(s)(kx2^m)^{-s}}{\zeta(s)} ds \\ &= \lim_{k \rightarrow \infty} \left(-x \frac{d}{dx} \right) \prod_{j=1}^k \left(1 - \frac{x}{j} \frac{d}{dx} \right) \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} 2^m \mu(n) g(xkn2^m), \end{aligned}$$

which, in turn, yield (2.16). □

Transformation (2.15) can be generalized considering the following two-parametric family of functions

$$U_{k,m}(x) = \frac{1}{2\pi i} \int_c [(1 - 2^{1-s})\zeta(s)]^{k+1} \Gamma^{m+1}(s) x^{-s} ds, x > 0, k, m \in \mathbb{N}_0. \quad (2.17)$$

The case $k = m$ we denote by $U_k(x)$. The case $k = m = 0$ gives $U_0(x) = (e^x + 1)^{-1}$. One can express the kernel (2.17) in terms of the iterated Mellin convolution. Indeed, via (1.14) and simple calculations we obtain

$$U_{k,m}(x) = \sum_{n_1, n_2, \dots, n_{k-m}=1}^{\infty} (-1)^{\sum_{j=1}^{k-m} n_j - k + m} \times \int_{\mathbb{R}_+^m} \prod_{j=1}^m (e^{u_j} + 1)^{-1} \left(\exp \left(\frac{x n_1 n_2 \dots n_{k-m}}{u_1 u_2 \dots u_m} \right) + 1 \right)^{-1} \frac{du_1 du_2 \dots du_m}{u_1 u_2 \dots u_m}, \quad k > m, \quad (2.18)$$

$$U_{k,m}(x) \equiv U_k(x) = \int_{\mathbb{R}_+^k} \left(\exp \left(\frac{x}{u_1 u_2 \dots u_k} \right) + 1 \right)^{-1} \prod_{j=1}^k (e^{u_j} + 1)^{-1} \frac{du_j}{u_j}, \quad k = m, \quad (2.19)$$

$$U_{k,m}(x) = \int_{\mathbb{R}_+^m} \prod_{j=1}^{k+1} (e^{u_j} + 1)^{-1} \exp \left(- \sum_{j=k+2}^m u_j \right) \exp \left(- \frac{x}{u_1 u_2 \dots u_m} \right) \frac{du_1 \dots du_m}{u_1 u_2 \dots u_m}, \quad k < m. \quad (2.20)$$

Thus an analog of Theorem 4 will be

Theorem 5 *Let $f \in \mathcal{M}^{-1}(L_c)$ and $c_0 > 1$. Then the integral transformation*

$$g(x) = \int_0^\infty U_{k,m} \left(\frac{x}{t} \right) f(t) \frac{dt}{t}, \quad x > 0 \quad (2.21)$$

is a bijective map between spaces $\mathcal{M}^{-1}(L_c)$, $\mathcal{M}_{(m+1)/2, (m+1)(1/2-c_0)}^{-1}(L_c)$ and for all $x > 0$ the following inversion formula takes place

$$f(x) = \lim_{l \rightarrow \infty} \left(-x \frac{d}{dx} \right)^{m+1} \prod_{j=1}^l \left(1 - \frac{x}{j} \frac{d}{dx} \right)^{m+1} \times \sum_{j_1, \dots, j_k=0}^{\infty} \sum_{n_1, \dots, n_k=1}^{\infty} \prod_{i=1}^k 2^{j_i} \mu(n_i) g \left(x l^{m+1} \prod_{i=1}^k 2^{j_i} n_i \right).$$

3 Transformations of the Kontorovich-Lebedev type

The familiar Kontorovich-Lebedev transform (see for instance in [9], [14], [16]) is defined by

$$K_{i\tau}[f] = \int_0^\infty K_{i\tau}(x) f(x) dx, \quad \tau \in \mathbb{R}_+, \quad (3.1)$$

where the integral converges in an appropriate sense and $K_\nu(x)$, $\nu \in \mathbb{C}$, $x > 0$ is the modified Bessel function [1, Vol. 2] having the following integral representations

$$K_\nu(2\sqrt{x}) = \frac{1}{4\pi i} \int_{a-i\infty}^{a+i\infty} \Gamma \left(s + \frac{\nu}{2} \right) \Gamma \left(s - \frac{\nu}{2} \right) x^{-s} ds, \quad a > |\operatorname{Re} \nu|, \quad (3.2)$$

$$K_\nu(x) = \int_0^\infty e^{-x \cosh u} \cosh \nu u \, du. \quad (3.3)$$

The main goal of this section is to consider an analog of the Kontorovich-Lebedev transform (3.1) involving the kernel, which we will call the Macdonald-Lambert function $\mathcal{M}_\nu(x)$, represented by

$$\mathcal{M}_\nu(x) = \int_0^\infty \frac{\cosh \nu u \, du}{e^{x \cosh u} - 1}, \quad x > 0, \quad \nu \in \mathbb{C}. \quad (3.4)$$

Precisely, letting in (3.4) ν as a pure imaginary number, $\nu = i\tau$, $\tau > 0$ let us consider the following transformation

$$\mathcal{M}_{i\tau}[f] = \int_0^\infty \mathcal{M}_{i\tau}(x) f(x) dx, \quad \tau \in \mathbb{R}_+. \quad (3.5)$$

First we observe via (3.4), that $\mathcal{M}_{i\tau}(x)$ is a real-valued function. Moreover, due to (3.3) and elementary summation it can be represented by the following series of the modified Bessel functions

$$\mathcal{M}_{i\tau}(x) = \sum_{n=1}^\infty K_{i\tau}(nx), \quad x > 0, \quad (3.6)$$

where the corresponding change of the order of integration and summation is by virtue of the absolute and uniform convergence. Hence invoking the uniform inequality for the modified Bessel function [16]

$$|K_{i\tau}(x)| \leq e^{-r\tau} K_0(x \cos r), \quad r \in [0, \pi/2),$$

we have, accordingly, the estimate

$$|\mathcal{M}_{i\tau}(x)| \leq e^{-r\tau} \sum_{n=1}^\infty K_0(nx \cos r) = e^{-r\tau} \mathcal{M}_0(x \cos r). \quad (3.7)$$

Meanwhile, making a simple change of variable and shifting the vertical contour in (3.2) to the right into the half-plane $\text{Res} > 1$, we substitute it in (3.6). Then inverting again the order of integration and summation owing to the absolute and uniform convergence, we employ (1.13) to deduce the formula

$$\mathcal{M}_{i\tau}(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} 2^{s-2} \Gamma\left(\frac{s+i\tau}{2}\right) \Gamma\left(\frac{s-i\tau}{2}\right) \zeta(s) x^{-s} ds, \quad a > 1. \quad (3.8)$$

Reciprocally, taking the Mellin transform of the kernel $\mathcal{M}_{i\tau}(x)$, it yields

$$\int_0^\infty \mathcal{M}_{i\tau}(x) x^{s-1} dx = 2^{s-2} \Gamma\left(\frac{s+i\tau}{2}\right) \Gamma\left(\frac{s-i\tau}{2}\right) \zeta(s), \quad \text{Res} = c_0 > 1 \quad (3.9)$$

and one can justify the absolute convergence of the integral in (3.9) shifting a contour in (3.8) to the left and to the right from the line $\text{Res} = c_0$ in order to get the corresponding behavior near zero and infinity, respectively.

A relationship of (3.5) with the Kontorovich-Lebedev transform (3.1) is given by

Lemma 1. *Let $f \in \mathcal{M}^{-1}(L_c)$, $c_0 = 1 - a$, $a > 1$. Then for all $\tau \in \mathbb{R}_+$*

$$\mathcal{M}_{i\tau}[f] = K_{i\tau}[g], \quad (3.10)$$

where $g(x)$ is the series transformation (see (2.6))

$$g(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) f^*(1-s) x^{s-1} ds = \sum_{n=1}^\infty \frac{1}{n} f\left(\frac{x}{n}\right) \quad (3.11)$$

and the following equality holds

$$\mathcal{M}_{i\tau}[f] = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} 2^{s-2} \Gamma\left(\frac{s+i\tau}{2}\right) \Gamma\left(\frac{s-i\tau}{2}\right) \zeta(s) f^*(1-s) ds. \quad (3.12)$$

Proof. In fact, since via conditions of the theorem

$$\int_0^\infty |\mathcal{M}_{i\tau}(x)| x^{a-1} \int_{a-i\infty}^{a+i\infty} |f^*(1-s)| ds dx < \infty,$$

the proof of (3.12) is straightforward by substitution (3.9) into the right-hand side of (3.12) and inversion of the order of integration with the use of Fubini's theorem and (3.5). In the same manner we prove composition (3.10), where $g(x)$ can be represented by (3.11) similarly to (2.6). \square

The main result of this section is an inversion theorem for the Kontorovich-Lebedev like transformation (3.5). For a different class of such index transformations and their inversions we refer to [15]. Our method will be based on Sneddon's operational approach to invert the Kontorovich-Lebedev transform (3.1) (see [9], Ch. 6).

We have

Theorem 6. *Let $f \in \mathcal{M}^{-1}(L_c)$, $c_0 = 1 - a$, $a > 1$. Let $f^*(s)$ be analytic in the strip $\text{Res} \in [1 - a, 1 + a]$, $a > 1$, $f^*(0) = 0$ and $\zeta(-c_0 - it)f^*(1 + c_0 + it) \in L_1(\mathbb{R}) \cap L_p(\mathbb{R})$, $p > 1$ for all $c_0 \in [-a, a]$. If $\mathcal{M}_{i\tau}[f] \in L_1(\mathbb{R}_+; \tau e^{\pi\tau} d\tau)$, then for almost all $x \in \mathbb{R}_+$ the following inversion formula holds*

$$xf(x) = \int_0^\infty \tau \sinh \pi\tau \hat{\mathcal{M}}_{i\tau}(x) \mathcal{M}_{i\tau}[f] d\tau, \quad x > 0, \quad (3.13)$$

where

$$\hat{\mathcal{M}}_{i\tau}(x) = \frac{2^{3/2}}{\sqrt{x} \tau \sinh(\pi\tau/2)} \sum_{n=1}^\infty \frac{\mu(n)}{n^{3/2}} \left(1 + \frac{4\pi^2}{x^2 n^2}\right)^{1/4} P_{(i\tau-1)/2}^{1/2} \left(1 + \frac{8\pi^2}{x^2 n^2}\right). \quad (3.14)$$

Proof. We begin substituting integral representation (3.4) with $\nu = i\tau$ into (3.5) and changing the order of integration by Fubini's theorem. It is indeed allowed via conditions of Lemma 1 and the estimate

$$\begin{aligned} \int_0^\infty du \int_0^\infty \frac{|f(x)| dx}{e^x \cosh u - 1} &\leq \frac{1}{2\pi} \int_0^\infty du \int_0^\infty \frac{x^{a-1}}{e^x \cosh u - 1} dx \int_{a-i\infty}^{a+i\infty} |f^*(1-s)| ds \\ &= \frac{1}{2\pi} \int_0^\infty \frac{du}{\cosh^a u} \int_0^\infty \frac{x^{a-1}}{e^x - 1} dx \int_{a-i\infty}^{a+i\infty} |f^*(1-s)| ds < \infty, \quad a > 1. \end{aligned}$$

Consequently,

$$\mathcal{M}_{i\tau}[f] = \int_0^\infty \cos \tau u \int_0^\infty \frac{f(x)}{e^x \cosh u - 1} dx du. \quad (3.15)$$

Hence as we see in the above estimate the inner integral with respect to x is an integrable function by u . Moreover, inequality (3.7) and conditions of the theorem guarantee that $\mathcal{M}_{i\tau}[f] \in L_1(\mathbb{R}_+)$. Thus inverting the cosine Fourier transform in (3.15) we arrive at the equality

$$\frac{2}{\pi} \int_0^\infty \mathcal{M}_{i\tau}[f] \cos \tau u d\tau = \int_0^\infty \frac{f(x) dx}{e^x \cosh u - 1},$$

or after simple substitution $v = \cosh u$ it becomes

$$\frac{2}{\pi} \int_0^\infty \mathcal{M}_{i\tau}[f] \cos \left(\tau \log \left(v + \sqrt{v^2 - 1} \right) \right) d\tau = \int_0^\infty \frac{f(x)}{e^{xv} - 1} dx, \quad v > 1. \quad (3.16)$$

A differentiation under integral sign in (3.16) with respect to v is still allowed by virtue of the absolute and uniform convergence of the corresponding integrals. Precisely, in its left hand-side it is owing to inequality (3.7) and in the right- hand side by the inequality

$$\int_0^\infty \frac{x|f(x)|e^{vx}}{(e^{xv}-1)^2}dx \leq \frac{1}{2\pi} \int_0^\infty \frac{x^a e^x}{(e^x-1)^2}dx \int_{a-i\infty}^{a+i\infty} |f^*(1-s)ds| < \infty, \quad a > 1.$$

Thus (3.16) yields

$$\begin{aligned} \frac{2}{\pi} \int_0^\infty \tau \mathcal{M}_{i\tau}[f] \frac{\sin(\tau \log(v + \sqrt{v^2-1}))}{\sqrt{v^2-1}} d\tau &= \int_0^\infty \frac{x f(x) e^{xv}}{(e^{xv}-1)^2} dx \\ &= \int_0^\infty x f(x) \sum_{n=1}^\infty n e^{-xvn} dx. \end{aligned}$$

But

$$\int_0^\infty x |f(x)| \sum_{n=1}^\infty n e^{-xvn} dx \leq \frac{\zeta(a)\Gamma(a+1)}{2\pi v^{a+1}} \int_{a-i\infty}^{a+i\infty} |f^*(1-s)ds|, \quad a > 1.$$

Therefore, an interchange of the order of integration and summation in the right-hand side of latter equality is allowed and after a simple change of variables we come out with

$$\frac{2}{\pi} \int_0^\infty \tau \mathcal{M}_{i\tau}[f] \frac{\sin(\tau \log(v + \sqrt{v^2-1}))}{\sqrt{v^2-1}} d\tau = \int_0^\infty e^{-vx} x \sum_{n=1}^\infty \frac{1}{n} f\left(\frac{x}{n}\right) dx. \quad (3.17)$$

Meanwhile, appealing to the identity (cf. [9, p. 359])

$$\frac{\sin(\tau \log(v + \sqrt{v^2-1}))}{\sqrt{v^2-1}} = \frac{1}{\pi} \sinh \pi \tau \int_0^\infty e^{-vx} K_{i\tau}(x) dx,$$

and via condition of the theorem $\mathcal{M}_{i\tau}[f] \in L_1(\mathbb{R}_+; \tau e^{\pi\tau} d\tau)$, we substitute the latter integral into (3.17) and change the order of integration. Then canceling the Laplace transform due to the uniqueness theorem for Laplace transform of integrable functions [11], we arrive for almost all $x > 0$ at the equality

$$\frac{2}{\pi^2} \int_0^\infty \tau \sinh \pi \tau K_{i\tau}(x) \mathcal{M}_{i\tau}[f] d\tau = \sum_{n=1}^\infty \frac{x}{n} f\left(\frac{x}{n}\right). \quad (3.18)$$

However, the right-hand side of (3.18) is given by the integral (3.11), which becomes after a simple change of variables as

$$\sum_{n=1}^\infty \frac{x}{n} f\left(\frac{x}{n}\right) = \frac{1}{2\pi i} \int_{-a-i\infty}^{-a+i\infty} \zeta(-s) f^*(1+s) x^{-s} ds.$$

Moreover, conditions of the theorem allow to shift the contour to the right by the Cauchy theorem and write for each $x > 0$

$$\frac{1}{2\pi i} \int_{-a-i\infty}^{-a+i\infty} \zeta(-s) f^*(1+s) x^{-s} ds = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(-s) f^*(1+s) x^{-s} ds. \quad (3.19)$$

Indeed, since $f^*(0) = 0$ and $f^*(s)$ is analytic in the strip $\text{Res} \in [1-a, 1+a]$, $a > 1$, we have that the limit of the product $\zeta(-s)f^*(1+s)$ exists when $s \rightarrow -1$ and it is analytic in the strip $\text{Res} \in [-a, a]$. Further as in [11], p. 125 the condition $\zeta(-c_0 - it)f^*(1+c_0 + it) \in L_p(\mathbb{R})$, $p > 1$ for all $c_0 \in [-a, a]$ implies that

$\zeta(-s)f^*(1+s)x^{-s}$, $s = c_0 + it$ goes to zero when $|t| \rightarrow \infty$ uniformly for $-a + \varepsilon \leq c_0 \leq a - \varepsilon$ for any small fixed positive ε . Therefore (3.19) holds. Returning to (3.18) and accounting (3.2) with Fubini's theorem, which is applicable under integrability condition on $\mathcal{M}_{i\tau}[f]$, it becomes

$$\begin{aligned} & \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} 2^{s-2} x^{-s} \left[\frac{2}{\pi^2} \int_0^\infty \tau \sinh \pi \tau \Gamma\left(\frac{s+i\tau}{2}\right) \Gamma\left(\frac{s-i\tau}{2}\right) \mathcal{M}_{i\tau}[f] d\tau \right] ds \\ &= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(-s) f^*(1+s) x^{-s} ds. \end{aligned} \quad (3.20)$$

Canceling the inverse Mellin transform from both sides of (3.20), because the integrands are L_1 -functions and dividing by $\zeta(-s)$, we obtain

$$f^*(1+s) = \frac{2^{s-1}}{\zeta(-s)\pi^2} \int_0^\infty \tau \sinh \pi \tau \Gamma\left(\frac{s+i\tau}{2}\right) \Gamma\left(\frac{s-i\tau}{2}\right) \mathcal{M}_{i\tau}[f] d\tau.$$

Hence taking the inverse Mellin transform over $(b-i\infty, b+i\infty)$, $1 < b < 2$ from both sides of the latter equality, which is possible owing to integrability conditions, we deduce inversion formula (3.13), where

$$\hat{\mathcal{M}}_{i\tau}(x) = \frac{1}{\pi^3 i} \int_{b-i\infty}^{b+i\infty} 2^{s-2} \Gamma\left(\frac{s+i\tau}{2}\right) \Gamma\left(\frac{s-i\tau}{2}\right) \frac{x^{-s}}{\zeta(-s)} ds, x > 0.$$

To complete the proof, we will show that the kernel $\hat{\mathcal{M}}_{i\tau}(x)$ can be written in the form (3.14). To do this we appeal to the functional equation (1.12) for the Riemann zeta-function and duplication formula for the Euler gamma - function. Thus it gives

$$\hat{\mathcal{M}}_{i\tau}(x) = -\frac{1}{2\pi^{5/2}i} \int_{b/2-i\infty}^{b/2+i\infty} \frac{\Gamma\left(s + \frac{i\tau}{2}\right) \Gamma\left(s - \frac{i\tau}{2}\right) \Gamma(s) \Gamma(1-s)}{\zeta(1+2s) \Gamma(s+1/2) \Gamma(1+s)} (x/2\pi)^{-2s} ds. \quad (3.21)$$

In the mean time the Parseval identity for the Mellin transform [11] and relations (8.4.19.1), (8.4.23.27) in [5], Vol. 3 lead to the equality

$$\begin{aligned} & -\frac{1}{2\pi^{5/2}i} \int_{b/2-i\infty}^{b/2+i\infty} \frac{\Gamma\left(s + \frac{i\tau}{2}\right) \Gamma\left(s - \frac{i\tau}{2}\right) \Gamma(s) \Gamma(1-s)}{\zeta(1+2s) \Gamma(s+1/2) \Gamma(1+s)} (x/2\pi)^{-2s} ds \\ &= -\frac{2}{\pi^2} \int_0^\infty K_{i\tau/2}^2\left(\frac{xy}{4\pi}\right) J_1(y) dy, \end{aligned}$$

where $J_1(y)$ is the Bessel function of the first kind [1], Vol. II. But the latter integral is calculated in [5], Vol. 2, relation (2.16.43.2) and we obtain the result

$$-\frac{2}{\pi^2} \int_0^\infty K_{i\tau/2}^2\left(\frac{xy}{4\pi}\right) J_1(y) dy = \frac{2^{3/2}}{\sqrt{x} \tau \sinh(\pi\tau/2)} \left(1 + \frac{4\pi^2}{x^2}\right)^{1/4} P_{(i\tau-1)/2}^{1/2}\left(1 + \frac{8\pi^2}{x^2}\right), \quad (3.22)$$

where $P_\nu^d(z)$ is the associated Legendre function [1], Vol. I. Hence, returning to (3.21) and combining with series (1.5), we substitute it inside the integral. Then changing the order of integration and summation via the absolute convergence and appealing to (3.22), we come out with (3.14). \square

4 Salem's type equivalences to the Riemann hypothesis

In 1953 Salem [7] proved that the Riemann hypothesis is true, i.e. the Riemann zeta-function $\zeta(s)$ is free of zeros in the strip $1/2 < \text{Re } s < 1$ is equivalent to the fact, that the homogeneous integral equation

$$\int_0^\infty \frac{y^{\delta-1}}{e^{xy} + 1} h(y) dy = 0, \quad x > 0, \quad \frac{1}{2} < \delta < 1, \quad (4.1)$$

has no nontrivial solutions in the space of bounded measurable functions on \mathbb{R}_+ . But after a simple change of variable this equation becomes (2.15), where $g(x) = 0$ and $f(t) = t^{-\delta} h(1/t)$. Therefore reciprocities (2.9), (2.10) and (2.15), (2.16) lead to

Corollary 1. *Let $h(x)$ be a solution of homogeneous equation (4.1) such that $x^{-\delta} h(1/x) \in \mathcal{M}^{-1}(L_c)$, $c_0 > 1$, $1/2 < \delta < 1$. Then $h(x) \equiv 0$.*

Proof. Indeed, there exists a function $h_\delta^*(s) \in L_1(c)$ such that

$$x^{-\delta} h\left(\frac{1}{x}\right) = \frac{1}{2\pi i} \int_c h_\delta^*(s) x^{-s} ds, \quad x > 0.$$

Hence

$$|h(x)| \leq \frac{x^{c_0-\delta}}{2\pi} \int_c |h_\delta^*(s)| ds$$

and since $c_0 > \delta$, we have that $h(x)$ is continuous on \mathbb{R}_+ and $h(x) = o(1)$, $x \rightarrow 0$. Applying inversion formula (2.16) with $g = 0$ we get the result. \square

Let us prove the following equivalence to the Riemann hypothesis of the Salem type.

Theorem 7. *The Riemann hypothesis is true, if and only if for any bounded measurable function $f(x)$ on \mathbb{R} satisfying integral equation*

$$\int_{\mathbb{R}^2} \frac{e^{-\delta u} f(u)}{(e^{x-u-t} + 1)(e^{et} + 1)} du dt = 0, \quad \frac{1}{2} < \delta < 1, \quad (4.2)$$

for all $x \in \mathbb{R}$ it follows that f is zero almost everywhere.

Proof. Calling again (1.14) and properties of the Mellin transform and its convolution [11], [14] it is not difficult to derive the equality

$$[(1 - 2^{1-s})\zeta(s)\Gamma(s)]^2 = \int_0^\infty t^{s-1} \int_0^\infty \frac{du}{u(e^{t/u} + 1)(e^u + 1)} dt, \quad \text{Re } s > 0. \quad (4.3)$$

On the other hand, the reciprocal inversion of the Mellin transform yields

$$\int_0^\infty \frac{du}{u(e^{x/u} + 1)(e^u + 1)} = \frac{1}{2\pi i} \int_{\mu-i\infty}^{\mu+i\infty} [(1 - 2^{1-s})\zeta(s)\Gamma(s)]^2 x^{-s} ds. \quad (4.4)$$

The left-hand side of (4.4) is positive and via (4.3)

$$\int_0^\infty \int_0^\infty \frac{t^{\delta-1} du dt}{u(e^{t/u} + 1)(e^u + 1)} = [(1 - 2^{1-\delta})\zeta(\delta)\Gamma(\delta)]^2,$$

which after a simple change of variables is equivalent to the condition

$$\int_{\mathbb{R}^2} \frac{e^{\delta y} du dy}{(e^{e^{y-u}} + 1)(e^{e^u} + 1)} < \infty.$$

Hence following as in [7] Wiener's ideas about an equivalence of the completeness in $L_1(\mathbb{R})$ of translations

$$e^{\delta(x-y)} \int_{\mathbb{R}} \frac{du}{(e^{e^{x-y-u}} + 1)(e^{e^u} + 1)}, \quad x \in \mathbb{R}$$

and the absence of zeros of $[(1 - 2^{1-s})\zeta(s)\Gamma(s)]^2$, i.e. zeros of $\zeta(s)$ in the critical strip $1/2 < \operatorname{Re} s < 1$, we complete the proof. \square

Remark 1. Reminding integral representation (3.3) of the modified Bessel function and invoking identity (1.4), one can write equality (4.4) in the form

$$\frac{1}{2} \int_0^\infty \frac{du}{u(e^{x/u} + 1)(e^u + 1)} = \sum_{n=1}^\infty d(n) [K_0(2\sqrt{nx}) - 4K_0(2\sqrt{2nx}) + 4K_0(4\sqrt{nx})]. \quad (4.5)$$

Hence substituting (4.5) into (4.2), we change the order of integration and summation via absolute and uniform convergence since (see Section 1) $d(n) = O(n^\varepsilon)$, $\varepsilon > 0$, $n \rightarrow \infty$. Consequently, Theorem 7 can be reformulated as

Theorem 8. *The Riemann hypothesis is true, if and only if for any bounded measurable function $f(x)$ on \mathbb{R} and all $x \in \mathbb{R}$ the equation*

$$\sum_{n=1}^\infty d(n) [(\mathcal{K}_n f)(x) - 4(\mathcal{K}_{2n} f)(x) + 4(\mathcal{K}_{4n} f)(x)] = 0,$$

where

$$(\mathcal{K}_n f)(x) = \int_{-\infty}^\infty e^{-\delta u} K_0 \left(2\sqrt{n} e^{(x-u)/2} \right) f(u) du, \quad \frac{1}{2} < \delta < 1,$$

is the Meijer type convolution transform [3], has no nontrivial solutions.

Finally a class of Salem's type equivalences to the Riemann hypothesis is given by

Theorem 9. *Let $k, m \in \mathbb{N}_0$, $k \leq m$ and the kernel $U_{k,m}(x)$, $x > 0$ is defined by formulas (2.19), (2.20), correspondingly. The Riemann hypothesis is true, if and only if for any bounded measurable function $f(x)$ on \mathbb{R} satisfying integral equation*

$$\int_{\mathbb{R}} e^{-\delta u} U_{k,m}(e^{x-u}) f(u) du = 0, \quad \frac{1}{2} < \delta < 1, \quad (4.6)$$

for all $x \in \mathbb{R}$ it follows that f is zero almost everywhere.

Proof. Employing inversion formula (1.15) of the Mellin transform, we derive, reciprocally, from (2.17)

$$[(1 - 2^{1-s})\zeta(s)]^{k+1} \Gamma^{m+1}(s) = \int_0^\infty U_{k,m}(t) t^{s-1} dt, \quad \operatorname{Re} s > 0.$$

Moreover, $U_{k,m}(x)$, $x > 0$ is positive (see (2.19), (2.20)) and for $\delta \in (1/2, 1)$

$$\int_0^\infty U_{k,m}(t) t^{\sigma-1} dt = [(1 - 2^{1-\sigma})\zeta(\sigma)]^{k+1} \Gamma^{m+1}(\sigma).$$

This yields

$$\int_{\mathbb{R}} e^{\delta y} U_{k,m}(e^y) dy < \infty.$$

Hence as in Theorem 6 the completeness in $L_1(\mathbb{R})$ of translations

$$e^{\delta(x-y)} U_{k,m}(e^{x-y}), \quad x \in \mathbb{R}$$

is equivalent to the absence of zeros of $[(1 - 2^{1-s})\zeta(s)]^{k+1}\Gamma^{m+1}(s)$, i.e. zeros of $\zeta(s)$ in the critical strip $1/2 < \delta < 1$.

□

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S.Yakubovich
Department of Mathematics,
Faculty of Sciences,
University of Porto,
Campo Alegre st., 687
4169-007 Porto
Portugal
E-Mail: syakubov@fc.up.pt